FOURIER → PDE ↔ SERIE

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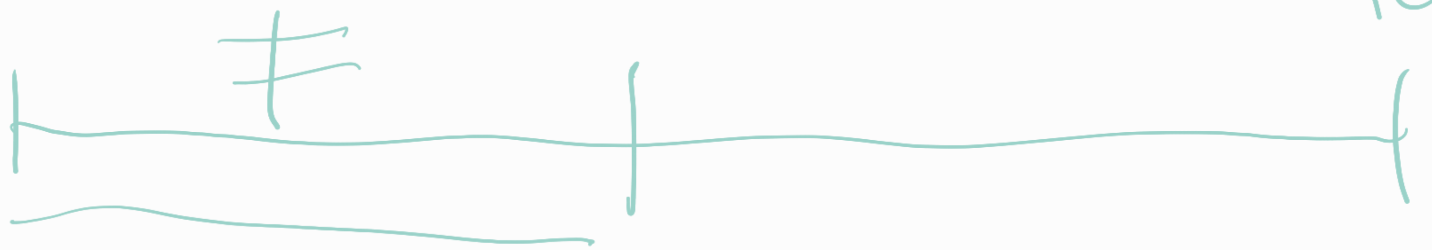

→ SV → Autofunzioni → Serie

→ osservabile → Temperature → Diffusion → Burgers → Trasf. Cole-Hopf

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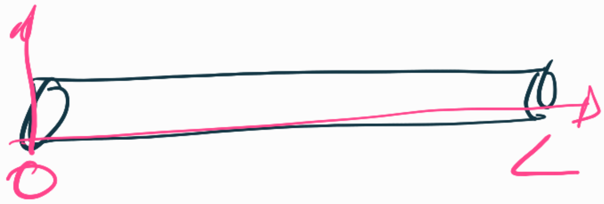
Fourier con ESTREMI ISOLATI:



$$\frac{\partial U}{\partial t} = K \frac{\partial^2 U}{\partial x^2} \qquad \text{PDE}$$

$$\frac{\partial U}{\partial x}(x=0) = \frac{\partial U}{\partial x}(x=L) = \phi \qquad \text{BC isolate}$$

$$U(x, t=\phi) = f(x) \qquad \text{CI}$$

Soluzione: SV → $U(x,t) = G(t)\,\phi(x)$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \Rightarrow \boxed{\phi(x)\frac{dG}{dt} = k\,G(t)\frac{d^2\phi}{dx^2}}$$

Divido m. a m. per $\frac{1}{\phi G k}$

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

ODE nel tempo

$$\boxed{\frac{dG}{dt} = -\lambda k G}$$

$$\boxed{G(t) = G_0 e^{-\lambda k t}}$$

nel tempo

ODE nello spazio

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

Nello spazio

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \to \begin{cases}\lambda = 0\\ \lambda > 0\\ \lambda < 0\end{cases}$$

Caso $\lambda = 0$

$$\frac{d^2\phi}{dx^2} = 0 \to \boxed{\phi(x) = C_1 x + C_2}$$

BCs: $\frac{d\varphi}{dx}(x=0)=\phi \rightarrow \boxed{C_1=0}$

$\frac{d\varphi}{dx}(x=L)=\phi \rightarrow \boxed{C_1=0}$ $\Bigg\}$ $\boxed{\varphi(x)=C_2}$

caso $\lambda>0$ $\frac{d^2\varphi}{dx^2}=-\lambda\varphi$

$$\boxed{\varphi(x)=A\sin(\sqrt{\lambda}x)+B\cos(\sqrt{\lambda}x)}$$

$$\frac{d\varphi}{dx}=A\sqrt{\lambda}\cos(\sqrt{\lambda}x)-B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

impongo le BCs $\rightarrow \frac{d\varphi}{dx}(x=0)=A\sqrt{\lambda}=0 \rightarrow \boxed{A=\phi}$

$\frac{d\varphi}{dx}(x=L)=\phi \Rightarrow -B\sqrt{\lambda}\sin(\sqrt{\lambda}L)=0$

$\rightarrow \sqrt{\lambda}L=n\pi \Rightarrow \boxed{\lambda_n=\left(\frac{n\pi}{L}\right)^2}$

contributo da $\lambda>0$ $\rightarrow \boxed{\varphi(x)=B_n\cos\left(\frac{n\pi}{L}x\right)}$

caso $\lambda < 0$

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \Rightarrow \varphi(x) = e^{\pm\sqrt{-\lambda}x}$$

$$\varphi(x) = \tilde{A}\sinh(\sqrt{-\lambda}\,x) + \tilde{B}\cosh(\sqrt{-\lambda}\,x)$$

$$\frac{d\varphi}{dx} = \tilde{A}(\sqrt{-\lambda})\cosh(\sqrt{-\lambda}\,x) + \tilde{B}(\sqrt{-\lambda})\sinh(\sqrt{-\lambda}\,x)$$

$\Rightarrow$ BC

$\Rightarrow \frac{d\varphi}{dx}(x=0) = 0 \Rightarrow \boxed{\tilde{A} = 0}$

$\Rightarrow \frac{d\varphi}{dx}(x=L) = 0 \Rightarrow \tilde{B}(\sqrt{-\lambda})\sinh(\sqrt{-\lambda}\cdot L) = 0 \Rightarrow \boxed{\tilde{B} = 0}$

$\nexists$ contributi del caso $\lambda < 0$ $\Leftarrow$

PRINCIPIO DI SOVRAPPOSIZIONE

$$U(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \phi$$

contributo di $\lambda = 0$ (fisicamente prioritario)

contributo da $\lambda > 0$

contributo da $\lambda < 0$

DETERMINARE i coefficienti

↳ studiare la C.I. $U(x, t=0) = f(x)$

espando

$$f(x) \sim \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\to f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

moltiplicare m.a.m. per $\cos\left(\frac{m\pi}{L}x\right)$
ed INTEGRO

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0 \cos\left(\frac{m\pi}{L}x\right)dx$$

$$+ \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}_{\frac{L}{2}\delta_{m,n}}$$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = A_0\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{m,n}$$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = A_0\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{m,n}$$

GUARDO $m=0$

$$\int_0^L f(x)\,dx = A_0\underbrace{\int_0^L dx}_{L} + \underbrace{\sum_{n=1}^{\infty}\frac{L}{2}\delta_{n,0}}_{=\emptyset}$$

$$\int_0^L f(x)\,dx = A_0 L \Rightarrow \boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

$$\lim_{t\to+\infty} U(x,t) = \lim_{t\to+\infty}\left[A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right]$$

$$= A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

Ortogonalità dei coseni

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases}$$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}_{n=m}$$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = A_m \underbrace{\int_0^L \cos^2\left(\frac{m\pi}{L}x\right)dx}_{L/2}$$

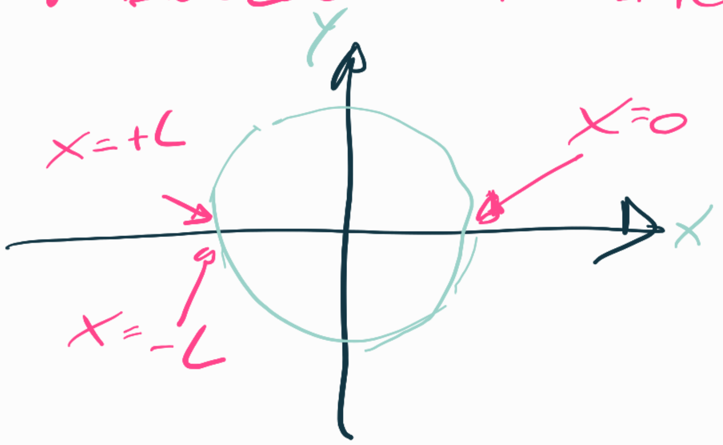
$$w \to w_x = \frac{\vec{w}\cdot\hat{i}}{\hat{i}\cdot\hat{i}}$$

$$\boxed{A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx}$$

CONDUZIONE DEL CALORE IN ANELLO CIRCOLARE

circonferenza $2L = 2\pi R$

$\to R = \frac{L}{\pi}$



$$x \in [-L, +L]$$

Richieste

- Sia continua la temperatura

$$u(x=+L, t) = u(x=-L, t)$$

- Sia continuo il flusso di calore

$$\frac{\partial u}{\partial x}(x=+L, t) = \frac{\partial u}{\partial x}(x=-L, t)$$

Voglio studiare allora

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \quad \text{con BC} \begin{cases} u(+L,t) = u(-L,t) \\ \frac{\partial u}{\partial x}(+L,t) = \frac{\partial u}{\partial x}(-L,t) \end{cases}$$

IC $u(x, t=0) = f(x)$

S.V. $\rightarrow$ $U(x,t) = G(t)\,\phi(x)$

PDE $\rightarrow$ 2 ODE

$\boxed{\text{Tempo}}$ $\dfrac{dG}{dt} = -\lambda k G \rightarrow \boxed{G(t) = G_0 e^{-\lambda k t}}$

$\boxed{\text{Spazio}}$ $\boxed{\dfrac{d^2\phi}{dx^2} = -\lambda\phi}$ Con $\begin{cases} \phi(-L) = \phi(+L) \\ \dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(+L) \end{cases}$ MIXED BCS

CASO $\lambda > 0$

$\phi(x) = C_1 \cos(\sqrt{\lambda}\,x) + C_2 \sin(\sqrt{\lambda}\,x)$

BC1 $\phi(-L) = \phi(+L)$ cosa implica?

$C_1 \cos(\sqrt{\lambda}\cdot(+L)) + C_2 \sin(\sqrt{\lambda}(+L)) =$

$= C_1 \cos(\sqrt{\lambda}\cdot(-L)) + C_2 \sin(\sqrt{\lambda}\cdot(-L))$

$\cos(-\sqrt{\lambda}L) = \cos(\sqrt{\lambda}L)$ PARI

$\sin(-\sqrt{\lambda}L) = -\sin(\sqrt{\lambda}L)$ DISPARI

allora posso scrivere la 1ª BC così:

$$\underbrace{C_1\cos(-L\sqrt{\lambda}) + C_2\sin(-L\sqrt{\lambda}) = C_1\cos(L\sqrt{\lambda}) + C_2\sin(L\sqrt{\lambda})}$$

$$\cancel{C_1\cos(L\sqrt{\lambda})} - C_2\sin(L\sqrt{\lambda}) = \cancel{C_1\cos(L\sqrt{\lambda})} + C_2\sin(L\sqrt{\lambda})$$

$$\Rightarrow \boxed{C_2\sin(\sqrt{\lambda}L) = \phi}$$

QUINDI

$$\sqrt{\lambda}L = n\pi \Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$$

studiamo ora la II BC (del flusso)

$$\frac{d\varphi}{dx}(x=-L) = \frac{d\varphi}{dx}(x=+L)$$

$$\frac{d\varphi}{dx} = \sqrt{\lambda}\left[-C_1\sin(\sqrt{\lambda}x) + C_2\cos(\sqrt{\lambda}x)\right]$$

$$-C_1 \sin\left[\sqrt{\lambda}\cdot(-L)\right] + C_2 \cos\left[\sqrt{\lambda}\cdot(-L)\right] =$$

$$= -C_1 \sin\left[\sqrt{\lambda}(+L)\right] + C_2 \cos\left[\sqrt{\lambda}(+L)\right]$$

$$\hookrightarrow \boxed{C_1 \sin\left(\sqrt{\lambda}\, L\right) = 0}$$

$\rightarrow$ DOB sia $C_1 \neq 0$, $C_2 \neq 0$

$$\phi(x) = \left\{ \cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right) \right\} \quad n \in \mathbb{N}$$

Sol: $\exists$ 2 famiglie infinite di soluzioni

$$\boxed{\begin{aligned} U_c(x,t) &= A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} \\ U_s(x,t) &= B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}} \quad \underline{\underline{\lambda > 0}}$$

CASO $\lambda = 0$

$$\frac{d^2\varphi}{dx^2} = 0$$

$$\varphi(x) = c_1 + c_2 x$$

BC1: $\varphi(-L) = \varphi(+L) \Rightarrow c_1 + c_2 L = c_1 - c_2 L \Rightarrow \boxed{c_2 = 0}$

BC2: $\frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(+L) \longrightarrow \boxed{c_2 = c_2 = 0}$

$\longrightarrow \boxed{\varphi(x) = c_1}$ $\Rightarrow$ è la costante nella sol. generale

CASO $\lambda < 0$: matematicamente

è banale ma fisicamente è ovvio...

La sol. generale è pronta e si legge:

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

per stimare i coefficienti $\rightarrow$ CI

espando $U(x,t=0) = f(x)$ in Base di Fourier

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

moltiplico m.a.m. per $\left\{\begin{matrix}\cos\left(\frac{m\pi}{L}x\right) \\ \sin\left(\frac{m\pi}{L}x\right)\end{matrix}\right\}$ ed integro

$$\int_{-L}^{+L} f(x) \begin{bmatrix}\cos\left(\frac{m\pi}{L}x\right) \\ \sin\left(\frac{m\pi}{L}x\right)\end{bmatrix} dx = \sum_{n=0}^{\infty} a_n \int_{-L}^{+L} \cos\left(\frac{n\pi}{L}x\right) \begin{bmatrix}\cos\left(\frac{m\pi}{L}x\right) \\ \sin\left(\frac{m\pi}{L}x\right)\end{bmatrix} dx$$

$$+ \sum_{n=0}^{\infty} b_n \int_{-L}^{+L} \sin\left(\frac{n\pi}{L}x\right) \begin{bmatrix}\cos\left(\frac{m\pi}{L}x\right) \\ \sin\left(\frac{m\pi}{L}x\right)\end{bmatrix} dx$$

questo mi da

$$\int_{-L}^{+L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx = a_m \int_{-L}^{+L} \cos^2\left(\frac{m\pi}{L}x\right)dx$$

$$\int_{-L}^{+L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx = b_m \int_{-L}^{+L} \sin^2\left(\frac{m\pi}{L}x\right)dx$$

QUINDI:

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\,dx$$

$$a_m = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx$$

$$b_m = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx$$

VALORI NUMERICI DEI COEFFICIENTI

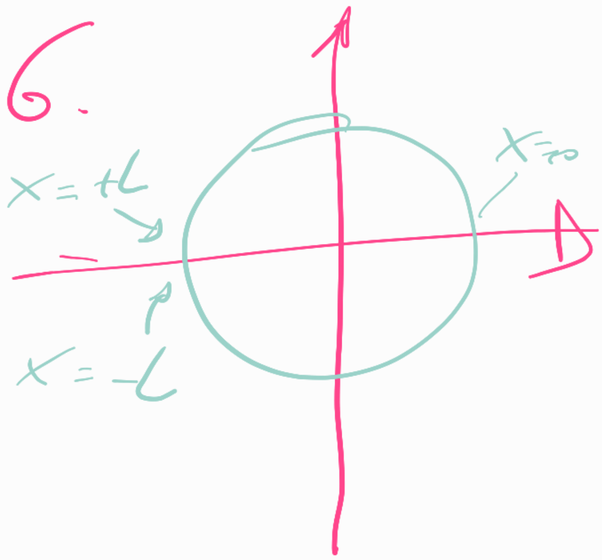
tutto questo vale perché:

$$\int_{-L}^{+L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & n\neq m \\ L & n=m\neq 0 \\ 2L & n=m=0 \end{cases}$$

$$\int_{-L}^{+L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & n\neq m \\ L & n=m\neq 0 \end{cases}$$

$$\int_{-L}^{+L} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = 0 \quad \forall n,m\in\mathbb{N}$$

ESERCIZIO 2.4.6.



$U(x,t)$ all'equilibrio

in entrambe i modi

cioè $\rightarrow \dot{U} = 0 \rightarrow \frac{d^2U}{dx^2} = 0$

$\rightarrow$ Risolvo $U(x,t) \rightarrow \lim_{t\to+\infty} U(x,t)$

PDE	BC	CI
$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$	$x\in[-L,+L]$ $U(-L,t) = U(+L,t)$ $\frac{\partial U}{\partial x}(-L,t) = \frac{\partial U}{\partial x}(+L,t)$	$U(x,0) = f(x)$

METODO UNO : ASSUMO STAZIONARIETÀ

$$\dot{U} = 0 \rightarrow \frac{d^2U}{dx^2} = 0 \rightarrow \boxed{U(x) = Ax + B}$$

$$U(-L) = U(+L) \Rightarrow \underbrace{-AL + B = +AL + B}$$

$$\boxed{U(x) = B} \Leftarrow \boxed{A = 0}$$

METODO DUE : risolvere la PDE e fare $\lim_{t \to \infty} U(x,t)$

$$U(x,t) = \phi(x)\, G(t) \Rightarrow \phi \frac{\partial G}{\partial t} = k G \frac{\partial^2 \phi}{\partial x^2}$$

Problema nel TEMPO

Tempo $\frac{dG}{dt} = -\lambda k G \rightarrow \boxed{G(t) = G_0 e^{-\lambda k t}}$

Spazio $\frac{d^2\phi}{dx^2} = -\lambda \phi$ — $\lambda > 0$, $\lambda = 0$, $\lambda < 0$

$$\phi(-L) = \phi(+L)$$

$$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(+L)$$

caso $\lambda = 0$

$$\phi(x) = \tilde{C}_1 x + \tilde{C}_2$$

$$\phi(-L) = \phi(+L) \Rightarrow \underbrace{\tilde{C}_1(-L) + \tilde{C}_2 = \tilde{C}_1(+L) + \tilde{C}_2}_{\boxed{\tilde{C}_1 = 0}} \rightarrow \boxed{\phi(x)_{\lambda=0} = \tilde{C}_2}$$

$$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(+L) \rightarrow \tilde{C}_1 = \tilde{C}_1$$

Caso $\lambda > 0$ $\qquad \frac{d^2\phi}{dx^2} = -\lambda\phi$

$$\phi(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

BCs: $\phi(+L) = \phi(-L) \Rightarrow \phi(+L) - \phi(-L) = \phi$

$$C_1 \cos[\sqrt{\lambda}\cdot(+L)] + C_2 \sin[\sqrt{\lambda}\cdot(+L)] =$$

$$C_1 \cos[\sqrt{\lambda}\cdot(-L)] + C_2 \sin[\sqrt{\lambda}\cdot(-L)]$$

BC

quindi: (ricordo: sin è odd e cos è even)

$$C_1\left\{\cancel{\cos[\sqrt{\lambda}(+L)]} - \cancel{\cos[\sqrt{\lambda}\cdot(-L)]}\right\} +$$

$$C_2\left\{\sin\left[\sqrt{\lambda}\cdot(+L)\right]-\sin\left[\sqrt{\lambda}(-L)\right]\right\}=0$$

$$2C_2\sin\left[\sqrt{\lambda}\cdot L\right]=0\rightarrow \boxed{\lambda_n=\left(\frac{n\pi}{L}\right)^2}$$

$$\boxed{\phi(x)=C_1\cos\left(\frac{n\pi}{L}x\right)+C_2\sin\left(\frac{n\pi}{L}x\right)}$$

caso $\lambda<0$

$$\phi(x)=A\cosh\left(\sqrt{-\lambda}\cdot x\right)+B\sinh\left(\sqrt{-\lambda}\cdot x\right)$$

$$BC_1:\ \phi(-L)=\phi(+L)$$

cosh → pari
sinh → dispari

$$\rightarrow \cancel{A\cosh\left[\sqrt{-\lambda}\cdot L\right]}+B\sinh\left[\sqrt{-\lambda}\cdot L\right]=$$

$$\cancel{A\cosh\left[\sqrt{-\lambda}\cdot(-L)\right]}+B\sinh\left[\sqrt{-\lambda}(-L)\right]$$

$$2B\sinh\left[\sqrt{-\lambda}\cdot L\right]=0\rightarrow \boxed{B=0}$$

BC 2 : $\frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(+L)$

$\hookrightarrow 2A \sinh\left[\sqrt{-\lambda}\, L\right] = 0 \rightarrow \boxed{\cancel{A} = 0}$

$\rightarrow \nexists$ contributo da $\lambda < 0$

φ ha 2 contributi, $\lambda = 0$ e $\lambda > 0$

$\hookrightarrow$ U x costruzione si legge

$$U(x,t) = \tilde{C}_2 + \sum_{n=1}^{\infty}\left[C_{1n}\cos\left(\frac{n\pi}{L}x\right) + C_{2n}\sin\left(\frac{n\pi}{L}x\right)\right]e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\boxed{\lim_{t\to+\infty} U(x,t) = \tilde{C}_2}$$

Per trovare il Direzello da Bevo?

$$U(x,t=0) = f(x) = \tilde{C}_2 + \sum_{n=1}^{\infty}\left[C_{1n}\cos\left(\frac{n\pi}{L}x\right) + C_{2n}\sin\left(\frac{n\pi}{L}x\right)\right]$$

$$\int_{-L}^{+L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_{-L}^{+L}\tilde{C}_2\cos\left(\frac{m\pi}{L}x\right)dx +$$

$$+\sum_{n=1}^{+\infty}\left[C_n\int \cos(n)\cos(m)+\varepsilon_n\int \sin(n)\cos(m)\right.$$

In Particolare $m=0$

$$\int_{-L}^{+L} f(x)\cdot 1\,dx = \tilde{C}_2\int_{-L}^{+L} 1\cdot dx + \phi$$

$$\boxed{\tilde{C}_2 = \frac{1}{2L}\int_{-L}^{+L} f(x)\,dx}$$

CENNI sulla Serie Di Fourier

$x \in [-L,+L]$

$$f(x) \simeq \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)}_{\text{SERIE DI FOURIER}}$$

Def 1

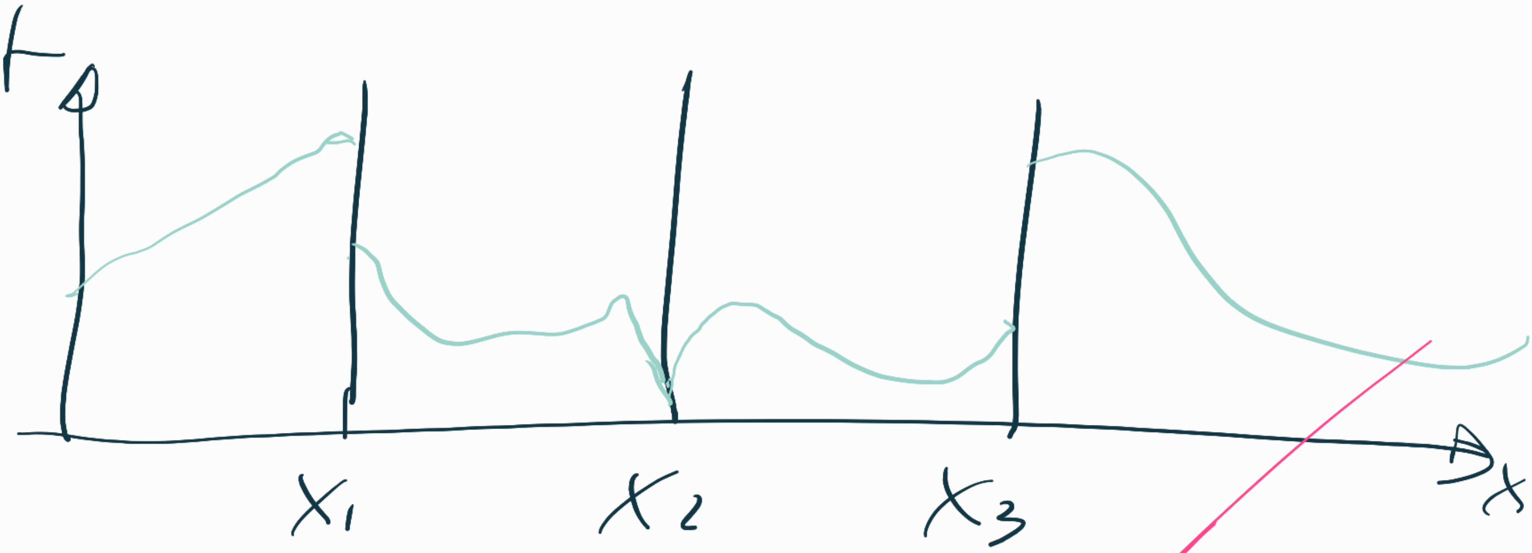
Coefficienti della SERIE Def 2

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\,dx$$

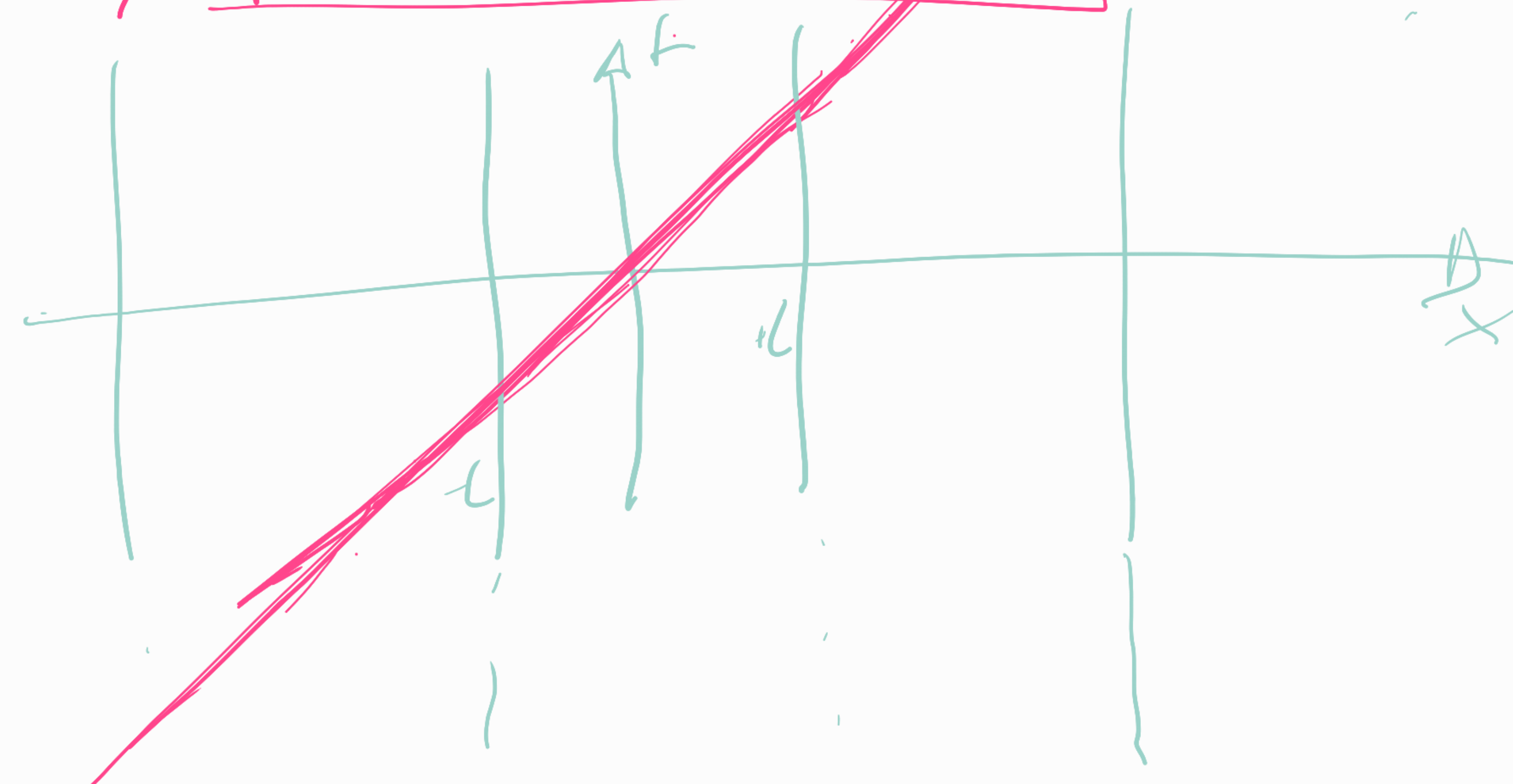
$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

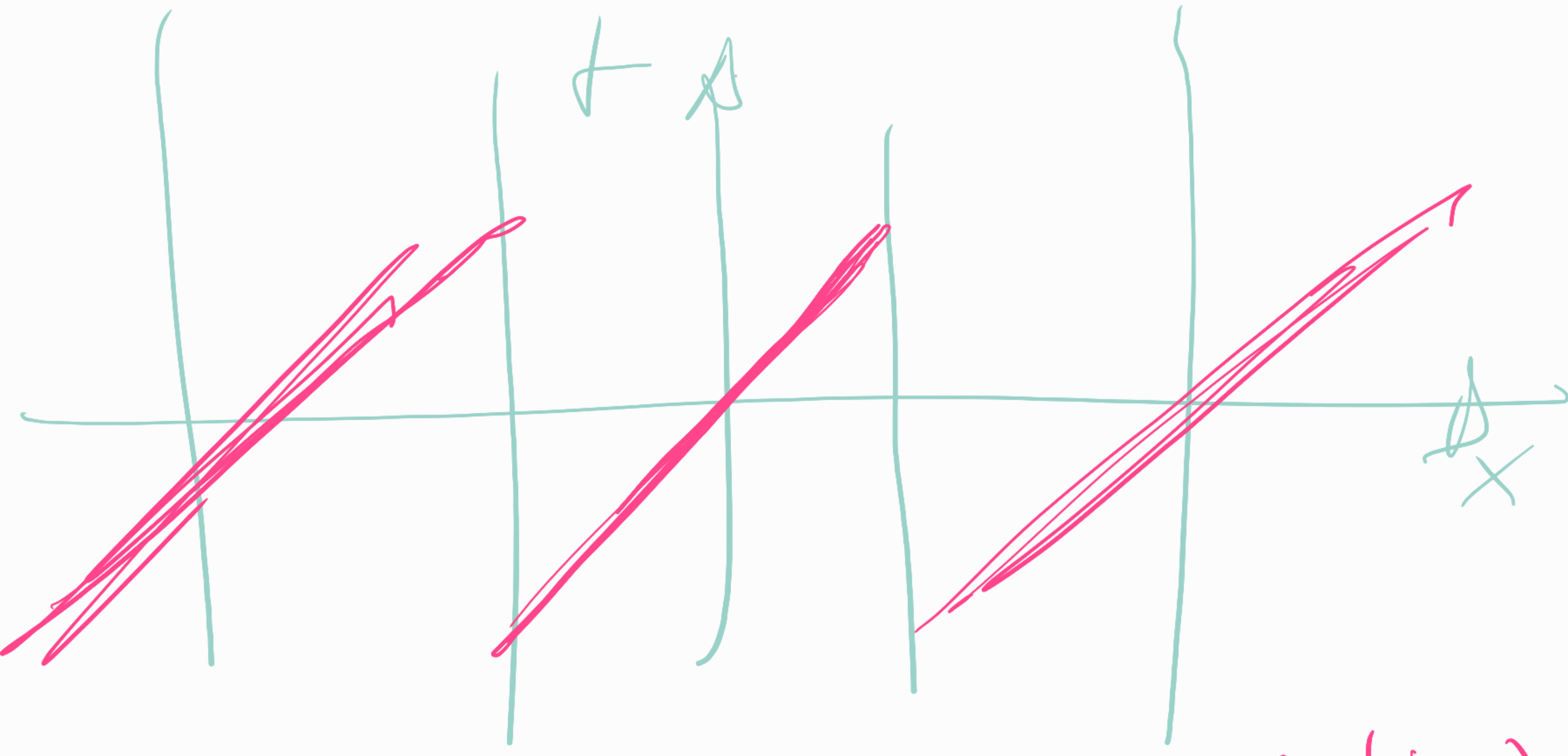
NB) $\left|\int_{-L}^{+L} f(x)\,dx\right| < +\infty$ Non calcolare nulla

def Funzioni continue a tratti



def) Periodica estendere





TEOREMA DI CONVERGENZA (di Fourier)

Se $f(x)$ è 1 fuc continua a tratti sull'intervallo $-L \le x \le +L$, allora la sua serie di Fourier converge a

1. all'estensione periodica di $f(x)$
2. alla media dei due limiti dove c'è il salto

$$\frac{1}{2}\left[f(x_+) + f(x_-)\right]$$