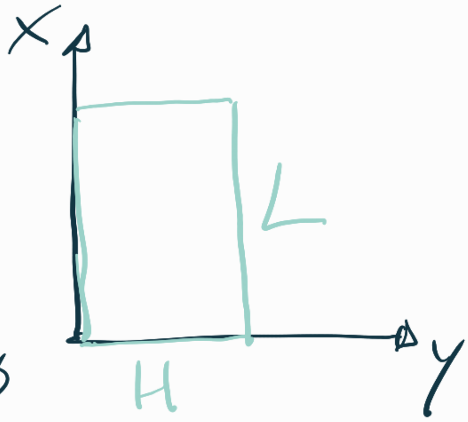
EQUAZIONE DI LAPLACE 2D

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$



BCs

$$U(x=0, y) = g(y) \qquad U(x, y=0) = 0$$

$$U(x=L, y) = 0 \qquad U(x, y=H) = 0$$

$U(x,y) = h(x)\,\varphi(y)$ ← metodo della sep. delle variabili

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \Longrightarrow \varphi \frac{d^2 h}{dx^2} + h \frac{d^2 \varphi}{dy^2} = 0 \quad \times$$

Dividendo $\frac{1}{h\varphi}$

$$\boxed{\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\varphi}\frac{d^2 \varphi}{dy^2} = \lambda}$$

INIZIAMO DA Y perché è 1 problema agli autovalori classico.

$$\frac{d^2\varphi}{dy^2} = -\lambda\varphi \quad \text{con BCs} \quad \boxed{\begin{array}{l}\varphi(\phi) = \phi \\ \varphi(H) = \phi\end{array}}$$

analizziamo i 3 casi
$\lambda = 0$
$\lambda < 0$
$\lambda > 0$

(1) caso $\lambda = 0$

$$\frac{d^2\varphi}{dy^2} = 0 \rightarrow \underline{\varphi(y) = C_1 y + C_2}$$

$\varphi(\phi) = \phi \rightarrow \underline{C_2 = 0}$

$\varphi(L) = \phi$
$\hookrightarrow C_1 L = \phi \rightarrow \underline{C_1 = 0}$

$\nexists$ contributi del caso $\lambda = 0$

caso $\lambda < 0$

$$\frac{d^2\varphi}{dy^2} = -\lambda\varphi$$

$$\hookrightarrow \varphi(y) = A\cosh(\sqrt{\lambda}\, y) + B\sinh(\sqrt{\lambda}\, y)$$

$\hookrightarrow \varphi(y=\phi)=\phi \rightarrow \boxed{A=0}$

$\varphi(y=H)=\phi \qquad B\sinh[\sqrt{\lambda}\,H]=\phi \rightarrow \boxed{B=0}$

Caso $\lambda>0$

$$\frac{d^2\varphi}{dy^2}=-3\varphi$$

$$\frac{d^2\varphi}{dy^2}=-\lambda\varphi$$

$$\varphi(y)=A\cos(\sqrt{\lambda}\,y)+B\sin(\sqrt{\lambda}\,y)$$

$\rightarrow BC_1 \quad \varphi(\phi)=\phi \rightarrow \boxed{A=0}$

$\rightarrow BC_2 \quad \varphi(H)=\phi \rightarrow B\sin(\sqrt{\lambda}\,H)=0$

$$\rightarrow \sqrt{\lambda}\,H=n\pi \rightarrow \boxed{\lambda_n=\left(\frac{n\pi}{H}\right)^2}$$

$$\varphi_n(y) = B_n \sin\left[\frac{n\pi}{H} y\right] \quad \forall \text{ di } Y$$

Proseguiamo con la variabile X

$$\frac{d^2 h}{dx^2} = \lambda_n h \quad \text{con} \quad h(L) = \phi, \; h(\phi) = g(y)$$

$$h(x) \sim e^{\pm\sqrt{\lambda_n} X}$$

$$h(x) = C_1 \cosh(\lambda_n X) + C_2 \sinh(\lambda_n X)$$

BCs $h(L) = \phi \Rightarrow \phi = C_1 \cosh(\lambda_n L) + C_2 \sinh(\lambda_n L)$

$$C_1 = -C_2 \,\mathrm{Tgh}(\lambda_n L)$$ metto questa espressione

$$\boxed{h(x) = C_2\left[-\mathrm{Tgh}(\lambda_n L)\cosh(\lambda_n X) + \sinh(\lambda_n X)\right]}$$

$$\Rightarrow h(x) = \tilde{C}_2\left[-\sinh(\lambda_n L)\cosh(\lambda_n X) + \cosh(\lambda_n L)\sinh(\lambda_n X)\right]$$

TRICK: $\sinh x \cosh y - \sinh y \cosh x = \sinh(x-y)$

$$\boxed{h(x) = \tilde{C}_2 \sinh\left[\lambda_n (x-L)\right]}$$

$U = h\varphi$

$A_n = B_n \tilde{C}_2$

$$\boxed{U(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \cdot \sinh\left[\left(\frac{n\pi}{H}\right)(x-L)\right]}$$

incognite da determinare

Non ho usato : $U(x=0,y) = g(y)$

$$U(0,y) = g(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left[\frac{n\pi}{H} (-L)\right]$$

$$\int_0^H g(y)\sin\left(\frac{m\pi}{H}y\right)dy =$$

$$= \sum_{n=1}^{\infty} A_n \sinh\left[\left(\frac{n\pi}{H}\right)(-L)\right] \underbrace{\int_0^H \sin\left(\frac{n\pi}{H}y\right)\sin\left(\frac{m\pi}{H}y\right)dy}_{\frac{H}{2}\delta_{nm}}$$

QUINDI

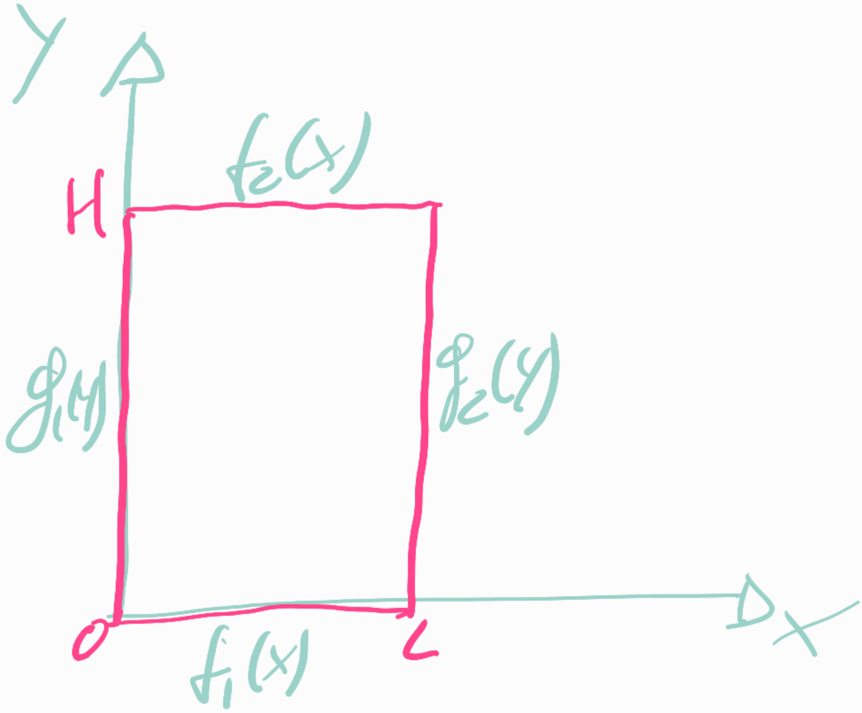
$$\int_0^H g(y)\sin\left(\frac{m\pi}{H}y\right)dy = \frac{H}{2} A_m \sinh\left[\left(\frac{m\pi}{H}\right)(-L)\right]$$

$$\hookrightarrow \boxed{A_m = \frac{2}{H\cdot\sinh\left[\left(\frac{m\pi}{H}\right)(-L)\right]}\int_0^L g(y)\sin\left(\frac{m\pi}{H}y\right)dy}$$ (2)

$$\boxed{U(x,y) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi}{H}y\right)\sinh\left[\left(\frac{m\pi}{H}\right)(x-L)\right]}$$ (1)

EQUAZIONE DI LAPLACE "COMPLETA"

PDE $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\phi$



BC$_1$ $U(x=\phi,y)=g_1(y)$

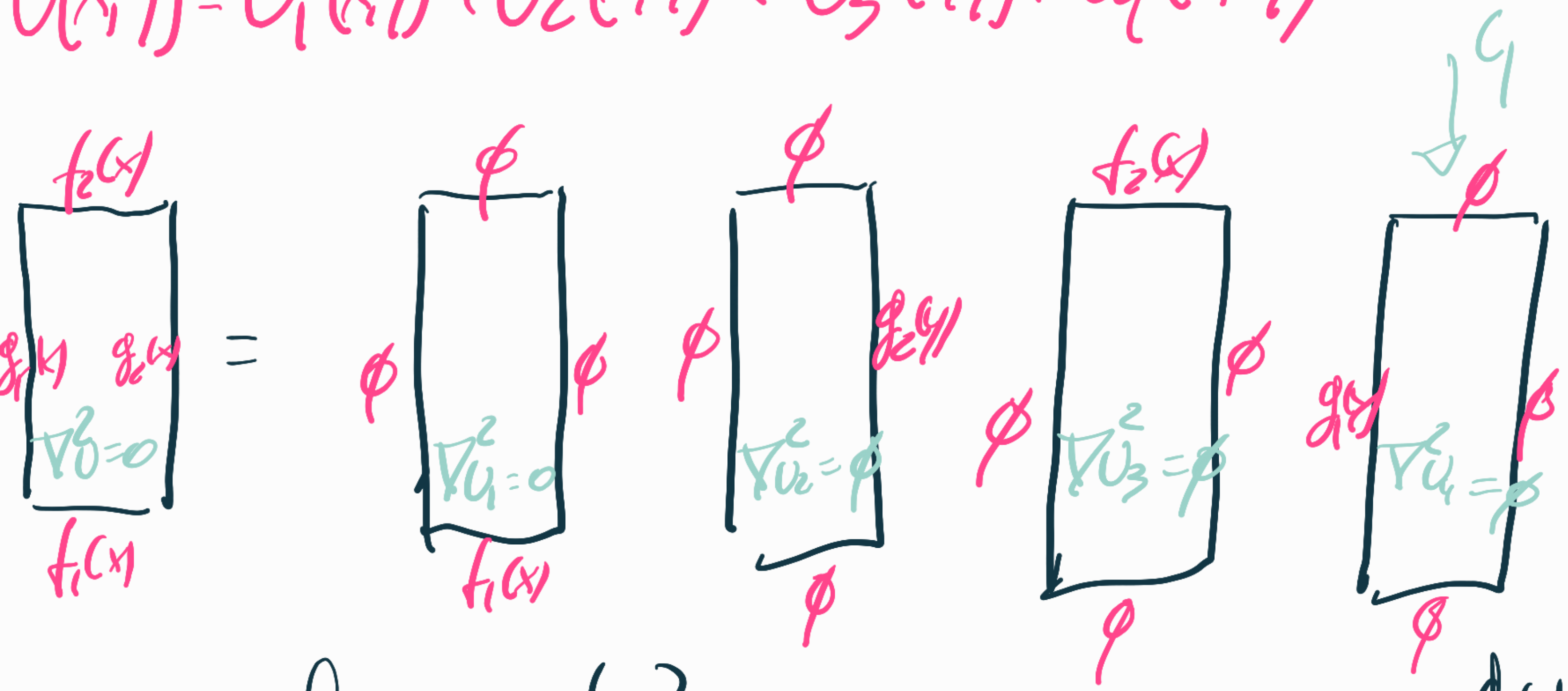
BC$_2$ $U(x=L,y)=g_2(y)$

BC$_3$ $U(x,y=0)=f_1(x)$

BC$_4$ $U(x,y=H)=f_2(x)$

TRUCCO: USO NATURA del Principio di sovrapposizione

$U(x,y)=U_1(x,y)+U_2(x,y)+U_3(x,y)+U_4(x,y)$



→ $x=0$ che succede? $U_1=\phi$ $U_2=\phi$ $U_3=\phi$ $U_4=g_1(y)$

Adesso risolvo 1 per 1 le 4 "sotto PDE".

Lo stesso per 1 sola

$$\frac{\partial^2 U_4}{\partial x^2} + \frac{\partial^2 U_4}{\partial y^2} = \phi \qquad U_4(0,y) = f_1(y),\; U_4(x,0) = \phi$$
$$U_4(L,y) = \phi,\; U_4(x,H) = \phi$$

↳ qui si deve usare Sep. di Variabili:

3 BCs BUONE

$$U_4(x,y) = h(x)\,\varphi(y) \qquad h(L) = \phi,\; \varphi(\phi) = \phi,\; \varphi(H) = \phi$$

$$\varphi \frac{d^2 h}{dx^2} + h \frac{d^2 \varphi}{dy^2} = \phi$$ Divido per $\frac{1}{h\varphi}$

$$\boxed{\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\varphi}\frac{d^2 \varphi}{dy^2} = \lambda}$$

Parte in X

$$\frac{d^2 h}{dx^2} = \lambda h$$ con solo $h(L) = \phi,\; h(0) = f_1(y)$.

Parte in Y

$$\frac{d^2\varphi}{dy^2} = -\lambda\varphi \quad \text{con} \quad \varphi(0) = \varphi(H) = 0$$

Risolvisto per la parte in φ (prob. ad autovalori standard)

$$\boxed{\varphi(y) = \sin\left(\frac{n\pi}{H}y\right), \quad \lambda_n = \left(\frac{n\pi}{H}\right)^2}$$

<u>Parte in x:</u>

$x \to x + L$

$$\frac{d^2h}{dx^2} = \lambda h = \left(\frac{n\pi}{H}\right)^2 h$$

$$\to A\cosh\left(\frac{n\pi}{H}x\right) + B\sinh\left(\frac{n\pi}{H}x\right) = h(x)$$

$$\to \boxed{h(x) = a_1\cosh\left[\frac{n\pi}{H}(x-L)\right] + a_2\sinh\left[\frac{n\pi}{H}(x-L)\right]}$$

$\to$ la sol. $\times$ u_4 è $u_4 = \varphi \cdot h$

$$U_4(x,y) = A \sin\left(\frac{n\pi}{H} y\right) \sinh\left[\frac{n\pi}{H}(x-L)\right]$$

$$\boxed{U_4(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left[\frac{n\pi}{H}(x-L)\right]}$$

$$g_1(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left[\frac{n\pi}{H}(x-L)\right]$$

i coefficienti sono $A_n \sinh\left[\frac{n\pi}{H}(x-L)\right]$

$$\to A_n \sinh\left[\left(\frac{n\pi}{H}\right)(-L)\right] = \frac{2}{H}\int_0^H g_1(y) \sin\left(\frac{n\pi}{H} y\right) dy$$

Soluzione generale x U_4 si scrive

$$U_4(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left[\left(\frac{n\pi}{H}\right)(x-L)\right]$$

$$A_n = \frac{2}{H \cdot \sinh\left[\left(\frac{n\pi}{H}\right)(-L)\right]} \int_0^H g_1(y) \sin\left(\frac{n\pi}{H} y\right) dy$$

↳ SOLUZIONE COMPLETA?

$$U(x,y) = U_1(x,y) + U_2(x,y) + U_3(x,y) + U_4(x,y)$$

ESERCITAZIONE

Ex 2.4.1. 4 casi per tutti vale

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{con} \quad \frac{\partial u}{\partial x}(x=0,t) = \frac{\partial u}{\partial x}(x=L,t) = \phi$$

caso a) $u(x,t=0) = \begin{cases} 0 & \text{se } x < L/2 \\ 1 & \text{se } x > L/2 \end{cases}$

caso b) $U(x,t=0) = 6 + 4\cos\left(\frac{3\pi}{L}x\right)$

caso c) $U(x,t=0) = -2\sin\left(\frac{\pi x}{L}\right)$

caso d) $U(x,t=0) = -3\cos\left(\frac{8\pi}{L}x\right)$

$$U(x,t) = G(t)\,\varphi(x) \rightarrow \varphi \frac{\partial G}{\partial t} = k G \frac{\partial^2 \varphi}{\partial x^2}$$

Divido $\frac{1}{k\varphi G}$ $\rightarrow$ $$\boxed{\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda}$$

$$\rightarrow \frac{dG}{dt} = -\lambda k G \rightarrow G(t) = C e^{-\lambda k t}$$

$$\rightarrow \frac{d^2\varphi}{dx^2} = -\lambda\varphi \quad \text{con} \quad \frac{d\varphi}{dx}(x=0) = \frac{d\varphi}{dx}(x=L) = \phi$$

STUDIO i 3 casi $\lambda = 0$, $\lambda > 0$, $\lambda < 0$

<u>Caso $\lambda = 0$</u> $\quad \varphi(x) = C_0 + C_1 x$

$$\frac{d\varphi}{dx} = C_1 = 0 \longrightarrow \boxed{\varphi(x) = C_0}$$

<u>Caso $\lambda > 0$</u>

$$\varphi(x) = A\sin(\sqrt{\lambda}\,x) + B\cos(\sqrt{\lambda}\,x)$$

$$\frac{d\varphi}{dx} = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

$$\to \frac{d\varphi}{dx}(x=0) = \phi \longrightarrow \boxed{A=0}$$

$$\to \frac{d\varphi}{dx}(x=L) = \phi \to -B\sqrt{\lambda}\sin(\sqrt{\lambda}L) = \phi$$

$$\to \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$$

$$\to \boxed{\varphi_n(x) = B_n \cos\left(\frac{n\pi}{L}x\right)}$$

<u>Caso $\lambda < 0$</u> : $\nexists$ contr. bvT.

$$\boxed{U(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$$f(x) = U(x,t=0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx$$

$$+\sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cdot\cos\left(\frac{m\pi}{L}x\right)dx}_{\left(\frac{L}{2}\right)\delta_{m,n}}$$

Cioè

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L A_0\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{m,n}$$

vero $\forall\, m$. scelgo $m=0$

$$\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

scelgo $m\neq 0$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \cancel{\frac{A_0 L}{m\pi}\sin\left(\frac{m\pi}{L}x\right)\Big|_0^L}$$

$$+\sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{m,n}$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx}$$

CASO a

$$f(x) = \begin{cases} 0 & 0 < x < \frac{L}{2} \\ 1 & \frac{L}{2} < x < L \end{cases}$$

$$A_0 = \frac{1}{L}\left[\int_0^{L/2} 0 \cdot dx + \int_{L/2}^{L} 1 \cdot dx\right] = \frac{1}{L}\frac{L}{2} = \frac{1}{2}$$

$$\boxed{A_0 = 1/2}$$

$$A_n = \frac{2}{L}\left[\cancel{\int_0^{L/2} 0 \cdot \cos\left(\frac{n\pi}{L}x\right)dx} + \int_{L/2}^{L} 1 \cdot \cos\left(\frac{n\pi}{L}x\right)dx\right]$$

PONGO $\frac{n\pi}{L}x = y \rightarrow dx = dy \cdot \frac{L}{n\pi}$

$$A_n = \frac{2}{L}\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big|_{L/2}^{L} =$$

$$\hookrightarrow A_n = \frac{2}{n\pi}\left[\cancel{\sin(n\pi)} - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$\hookrightarrow A_n = \frac{2}{n\pi}\left[-\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\begin{cases} 0 & n \text{ pari} \\ -1 & n = 1, 5, 9 \\ +1 & n = 3, 7, 11 \end{cases}$$

Soluzione totale

$$\boxed{U(x,t) = \frac{1}{2} + 2\sum_{n \text{ odd}} \left[\frac{-\sin\left(\frac{n\pi}{2}\right)}{n\pi}\right] \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

<u>caso b</u>

a: $f(x) = 6 + 4\cos\left(\frac{3\pi}{L}x\right)$

$A_0 = 6$, $A_3 = 4$, $A_n = 0$ se $n \neq (0,3)$

$$\boxed{U(x,t) = 6 + 4\cos\left(\frac{3\pi}{L}x\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}}$$

<u>caso c</u>

$$U(x, t=0) = -2\sin\left(\frac{\pi x}{L}\right)$$

$$A_0 = \frac{1}{L}\int_0^L \left[-2\sin\left(\frac{\pi x}{L}\right)\right]dx = \frac{-2}{L}\frac{L}{\pi}\left[-\cos\left(\frac{\pi x}{L}\right)\right]_\emptyset^L = \frac{-4}{\pi}$$

$$\boxed{A_0 = -4/\pi}$$

$$A_n = \frac{2}{L}\underbrace{\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)dx}_{I_n = \frac{2L}{\pi(1-n^2)}} = \begin{cases} \emptyset & \text{se } n \text{ dispari} \\ \dfrac{-8}{\pi(1-n^2)} & \text{se } n \text{ pari} \end{cases}$$

$$\boxed{\psi(x,t) = \frac{-4}{\pi}\left[1 + \sum_{n\ \text{pari}}^{2,4,6\ldots} \frac{2}{(1-n^2)}\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right]}$$

Ma perché $I_n \equiv \int_0^L \underline{\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)}dx = \frac{2L}{\pi(1-n^2)}$?

Trick: $\underline{\sin\alpha\cos\beta} = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$

$$I_n = \frac{1}{2}\left[\int_0^L \sin\left(\frac{\pi}{L}(1+n)x\right)dx + \int_0^L \sin\left[\frac{\pi}{L}(1-n)x\right]dx\right]$$

$$I_{n=1} = \frac{1}{2}\int_0^L \sin\left[\frac{2\pi}{L}x\right]dx = \frac{1}{2}\left(\frac{L}{2\pi}\right)\left(-\cos\left(\frac{2\pi x}{L}\right)\right)\Bigg|_0^L = 0$$

CONTRIBUTI $n > 1$

$$I_n = \frac{1}{2}\left[\frac{(-L)}{(1+n)\pi}\cos\left(\frac{\pi(1+n)x}{L}\right) + \frac{(-L)}{(1-n)\pi}\cos\left(\frac{\pi(1-n)x}{L}\right)\right]_0^L$$

$$= \frac{-L}{2\pi}\left\{\frac{1}{(1+n)}\left[\cos((1+n)\pi) - 1\right] + \frac{1}{(1-n)}\left[\cos((1-n)\pi) - 1\right]\right\}$$

$$= \frac{-L}{2\pi}\left\{\frac{1}{(1+n)}\left[(-1)^{1+n} - 1\right] + \frac{1}{(1-n)}\left[(-1)^{1-n} - 1\right]\right\}$$

NOTA: se n dispari fa tutto zero....

$$= \frac{-L}{2\pi}\left[\frac{-2}{1+n} + \frac{-2}{1-n}\right] = \left(\frac{-L}{2\pi}\right)\left[\frac{-2(1-\cancel{n}) - 2(1+\cancel{n})}{(1-n^2)}\right]$$

$$= \left(\frac{-L}{2\pi}\right)\left(\frac{-4}{(1-n^2)}\right) = \boxed{\frac{2L}{\pi(1-n^2)} = I_n}$$

CASO d

$$CI \quad U(x, t=0) = -3\cos\left(\frac{8\pi}{L}x\right)$$

$\rightarrow A_8 = -3, \quad A_n = 0 \text{ se } n \neq 8$

$$\boxed{U(x,t) = -3\cos\left(\frac{8\pi}{L}x\right) e^{-k\left(\frac{8\pi}{L}\right)^2 t}}$$

semantica di "Dimostra"

es 2.4.4 Si dimostri che $\nexists\ \lambda_n < 0$

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \quad \text{con} \quad \frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(L) = 0$$

risposta di $\int$ fisico

risposta di $\int$ matematico

è ovvio — $Q = 0$, $\nexists$ sorgenti; isolato

$$\varphi(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$$

$$\frac{d\varphi}{dx} = \alpha\left[C_1 e^{\alpha x} - C_2 e^{-\alpha x}\right]$$

$$\frac{d\varphi}{dx}(x=0) = 0 \rightarrow \alpha(C_1 - C_2) = 0 \rightarrow \boxed{C_1 = C_2}$$

Eulero

$$\frac{d\varphi}{dx}(x=L) = 0 \rightarrow \alpha\left(C_1 e^{\alpha L} - C_1 e^{-\alpha L}\right) = 2\alpha C_1 \sinh(\alpha L)$$

$$2\alpha C_1 \sinh(\alpha L) = 0 \rightarrow \boxed{C_1 = 0}$$

ES 2.4.3. Si risolva il seguente problema agli autovalori:

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \quad \text{con} \quad \begin{cases} \varphi(0) = \varphi(2\pi) \\ \dfrac{d\varphi}{dx}(0) = \dfrac{d\varphi}{dx}(2\pi) \end{cases}$$

CAMBIO Variabile: $x' = x - \pi$

Nella Nuova Variabile

$$\frac{d^2\varphi}{dx'^2} = -\lambda\varphi \quad \text{con} \quad \varphi(-\pi) = \varphi(\pi) \wedge \frac{d\varphi}{dx'}(-\pi) = \frac{d\varphi}{dx'}(+\pi)$$

→ Ci siamo ricondotti a teoria nota

Alla richiesta che Temp. e flusso siano continui:

Sol

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$\sin(nx - n\pi) = \sin(nx)\cdot\underbrace{\cos(n\pi)}_{(-1)^n} - \underbrace{\cos(nx)\sin(n\pi)}_{=0}$$

↳ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

↳ $\sin\left(\frac{n\pi}{L}x'\right) = (-1)^n \sin(nx)$

I classe di autof